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HAUSDORFF'S MOMENT PROBLEM AND EXPANSIONS IN LEGENDRE POLYNOMIALS--ETC(U)

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ABSTRACT

A new proof is given for Hausdorff's condition on a set of moments which determines when the function generating these moments is in L^2 . The proof uses Legendre polynomials and their discrete extensions found by Tchebichef. Then an extension is given to a weighted L^2 space using Jacobi polynomials and their discrete extensions.

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SIGNIFICANCE AND EXPLANATION

The paper describes a method of obtaining in terms of the moments approximations to the solutions of the finite moment problem

$$(1) \quad \int_0^1 f(x) x^v dx = \mu_v, \quad (v = 0, 1, 2, \dots).$$

In his paper [2] Hausdorff gave conditions on the moments μ_v for the problems (1) to have a solution $f(x)$ which is squares integrable. However, our approximations are constructed in terms of the coefficients c_v of the Legendre series expansion

$$f(x) \sim \sum_{v=0}^{\infty} c_v P_v(2x - 1), \quad (0 \leq x \leq 1),$$

where $P_v(x)$ are the Legendre polynomials. The main result is that Hausdorff's condition for a square integrable $f(x)$ are here expressed in terms of the c_v . This transition from the μ_v to the c_v is done by using a set of orthogonal polynomials on the discrete set $x = 0, x = 1, \dots, x = n$ due to Tchebychef.

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HAUSDORFF'S MOMENT PROBLEM AND EXPANSIONS IN LEGENDRE POLYNOMIALS

R. Askey, I. J. Schoenberg, and A. Sharma

1. Introduction. We refer to [3] for a description of the problem of Bellman, Kalaba and Lockett [1] of obtaining approximation to the inverse Laplace transform. They reduce the problem to the solution of the finite moment problem

$$(1.1) \quad \int_0^1 f(x)x^v dx = \mu_v, \quad (v = 0, 1, \dots, n-1)$$

and obtain approximations for $f(x)$ by applying Gauss' n -point quadrature formula to the integrals (1) and use numerical approximations to the inverse of the matrix of the system so obtained.

In [3] it is shown that the inverse of the Gauss matrix is not needed. Rather approximations to $f(x)$ are obtained if we determine the polynomial

$$(1.2) \quad R_{n-1}(x) = \sum_{v=0}^{n-1} c_v P_v(1-2x)$$

of degree $n-1$ which is the least square approximation to $f(x)$ in $[0,1]$ having moments $\mu_0, \mu_1, \dots, \mu_{n-1}$. The coefficients c_v in (1.2) are given by the lower triangular transformation

$$(1.3) \quad c_v = (2v+1) \sum_{i=0}^v (-1)^i \binom{v+i}{v} \binom{v}{i} \mu_i, \quad (v = 0, 1, \dots, n-1).$$

The numerical problem of Bellman, Kalaba and Lockett is thereby solved. However, this approach shows that the infinite problem in $[0,1]$

$$(1.4) \quad \int_0^1 f(x) x^v dx = \mu_v, \quad (v = 0, 1, 2 \dots \text{to infinity})$$

might be attacked in terms of the Legendre series expansion

$$(1.5) \quad f(x) \sim \sum_{v=0}^{\infty} c_v P_v(1-2x).$$

Hausdorff devoted to the problem (1.4) his famous paper [2] in which he showed the following:

A. The system

$$(1.6) \quad \int_0^1 x^v d\psi(x) = \mu_v, \quad (v = 0, 1, 2, \dots)$$

has a non-decreasing solution $\psi(x)$ if and only if

$$\Delta^n \mu_m = \mu_m - \binom{n}{1} \mu_{m+1} + \dots + (-1)^n \mu_{m+n} > 0$$

for $m, n > 0$.

B. The system (1.6) has a solution $\psi(x)$ of bounded variations in $[0, 1]$ if and only if

$$\sum_{v=0}^n \binom{n}{v} |\Delta^{n-v} \mu_v| = O(1) \quad \text{as } n \rightarrow \infty.$$

For a direct derivation of Hausdorff's conditions for A and B see [4].

C. The system (1.6) has a solution

$$\psi(x) = \int_0^x \varphi(x) dx$$

where $\varphi(x) \in L^p(0, 1)$ with $1 < p < \infty$, if and only if

$$(n+1)^{p-1} \sum_{v=0}^n \left\{ \binom{n}{v} |\Delta^{n-v} \mu_v| \right\}^p = O(1) .$$

A particular case of C is this ($p = 2$):

The moment problem

$$(1.7) \quad \int_0^1 f(x) x^v dx = \mu_v, \quad (v = 0, 1, 2, \dots)$$

has a solution $f(x) \in L^2(0,1)$ if and only if

$$(1.8) \quad S_n = (n+1) \sum_{v=0}^n \binom{n}{v}^2 (\Delta^{n-v} \mu_v)^2 = O(1) .$$

While Hausdorff's results A and B are most apt, it seems that the result (1.8) might be profitably deduced from the expansion (1.5). Since $\sqrt{2n+1} P_n(1-2x)$ are orthonormal, we derive from the Riesz-Fisher theorem and

$$f(x) \sim \sum_{v=0}^{\infty} \frac{c_v}{\sqrt{2v+1}} \sqrt{2v+1} P_v(1-2x)$$

the following: The moment problem (1.7) has a solution $f(x) \in L^2(0,1)$ if and only if

$$(1.9) \quad \sum_{v=0}^n \frac{c_v^2}{2v+1} = O(1) \text{ for all } n .$$

Formula (1.3) can be inverted and assumes the form

$$(1.10) \quad \mu_v = \frac{1}{(v+1) \binom{2v+1}{v}} \left\{ \binom{2v+1}{v} c_0 - \binom{2v+1}{v-1} c_1 + \dots + (-1)^v \binom{2v+1}{0} c_v \right\} .$$

Substituting (1.10) in (1.8) leads to

$$\begin{aligned}
s_1 &= c_0^2 + \frac{1}{9} c_1^2 \\
s_2 &= c_0^2 + \frac{1}{6} c_1^2 + \frac{1}{50} c_2^2 \\
s_3 &= c_0^2 + \frac{1}{5} c_1^2 + \frac{1}{25} c_2^2 + \frac{1}{245} c_3^2 .
\end{aligned}$$

These simple expressions were a surprise and suggested that

$$s_n = \sum_{v=0}^n a_{n,v} c_v^2$$

with $a_{n,v}$ given by a reasonably simple expression. This will be shown in the next section

2. Hausdorff Theorem via Orthogonal Polynomials

Using the notation introduced in the first section, we have

$$\begin{aligned}
\Delta^{n-v} \mu_v &= \int_0^1 x^v (1-x)^{n-v} f(x) dx \\
&= \sum_{k=0}^n c_k \int_0^1 P_k(1-2x) x^v (1-x)^{n-v} dx \\
&= \sum_{k=0}^n c_k \sum_{j=0}^k \frac{(-k)_j (k+1)_j}{(1)_j j!} \int_0^1 x^{v+j} (1-x)^{n-v} dx \\
(2.1) \quad &= \frac{\Gamma(v+1) \Gamma(n-v+1)}{\Gamma(n+2)} \sum_{k=0}^n c_k {}_3F_2 \left(\begin{matrix} -k, & k+1, & v+1 \\ & 1, & n+2 \end{matrix} ; 1 \right) .
\end{aligned}$$

The shifted factorial $(a)_n$ is defined by

$$(a)_n = \frac{\Gamma(n+a)}{\Gamma(a)} .$$

Murphy's formula for Legendre polynomials was used

$$(2.2) \quad P_n(x) = {}_2F_1 \left(\begin{matrix} -n, & n+1 \\ & 1 \end{matrix} ; \frac{1-x}{2} \right)$$

and the generalized hypergeometric function is defined by

$$(2.3) \quad {}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; t \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{t^n}{n!}.$$

Using (2.1) in S_n gives

$$(2.4) \quad S_n = \frac{1}{n+1} \sum_{k=0}^n \sum_{\ell=0}^n c_k c_{\ell} \sum_{v=0}^n {}_3F_2 \left(\begin{matrix} -k, k+1, v+1 \\ 1, n+2 \end{matrix}; 1 \right) {}_3F_2 \left(\begin{matrix} -\ell, \ell+1, v+1 \\ 1, n+2 \end{matrix}; 1 \right).$$

If this quadratic form is to be diagonal, then the following orthogonality relation must hold:

$$\sum_{x=0}^n {}_3F_2 \left(\begin{matrix} -k, k+1, x+1 \\ 1, n+2 \end{matrix}; 1 \right) {}_3F_2 \left(\begin{matrix} -\ell, \ell+1, x+1 \\ 1, n+2 \end{matrix}; 1 \right) = 0, \quad (0 \leq k \neq \ell \leq n).$$

Now

$$R_k(x) = {}_3F_2 \left(\begin{matrix} -k, k+1, x+1 \\ 1, n+2 \end{matrix}; 1 \right)$$

is a polynomial of degree n in x and it is relatively well-known that Tchebychef found a set of polynomials which are orthogonal on $x = 0, 1, \dots, n$ with respect to the uniform distribution (see [5], §2.8). This is what we want, but at first glance, it seems we do not have it, since Tchebychef's polynomials are usually given as

$$Q_k(x, n) = {}_3F_2 \left(\begin{matrix} -k, k+1, -x \\ 1, -n \end{matrix}; 1 \right), \quad x, k = 0, 1, \dots, n$$

and this does not seem to be the same as $R_k(x)$. However, there is a transformation formula which reconciles this difference,

$$(2.5) \quad {}_3F_2 \left(\begin{matrix} -k, a, b \\ c, d \end{matrix}; 1 \right) = \frac{(c-a)_k}{(c)_k} \cdot {}_3F_2 \left(\begin{matrix} -k, a, d-b \\ d, a+1-k-c \end{matrix}; 1 \right).$$

To obtain (2.5), write ([4], (4.1.3)) as an identity between hypergeometric series,

that is

$${}_2F_1\left(\begin{matrix} -k, a \\ c \end{matrix}; x\right) = \frac{(c-a)_k}{(c)_k} {}_2F_1\left(\begin{matrix} -k, a \\ a+1-k-c \end{matrix}; 1-x\right)$$

and integrate with respect to a beta distribution. Take $a = k + 1$, $b = x + 1$, $c = n + 2$ and $d = 1$ in (2.5) to get

$$(2.6) \quad R_k(x) = \frac{(n+1-k)_k}{(n+2)_k} \cdot {}_3F_2\left(\begin{matrix} -k, k+1, -x \\ 1, -n \end{matrix}; 1\right).$$

Using (2.6) above gives

$$S_n = \frac{1}{n+1} \sum_{k=0}^n \frac{(n+1-k)_k^2}{(n+2)_k^2} \cdot \sum_{x=0}^n [Q_k(x)]^2.$$

The orthogonality relation for $Q_k(x)$ is

$$\sum_{x=0}^n Q_k(x, n) Q_\ell(x, n) = \delta_{k\ell} \frac{(n+1)(n+2)_k}{(n+1-k)_k \cdot (2k+1)}$$

so

$$S_n = \sum_{k=0}^n \frac{c_k^2}{2k+1} \cdot \frac{(n+1-k)_k}{(n+2)_k}.$$

Since $(n+1-k)_k / (n+2)_k < 1$, $S_n < \sum_{k=0}^n c_k^2 / (2k+1)$, which proves one of the required inequalities.

Conversely, if $S_n = O(1)$, then

$$\begin{aligned} \sum_{k=0}^n \frac{c_k^2}{2k+1} &= \sum_{k=0}^n \frac{c_k^2}{(2k+1)} \cdot \frac{(n^2+1-k)_k}{(n^2+2)_k} \cdot \frac{(n^2+2)_k}{(n^2+1-k)_k} \\ &\leq \frac{(n^2+2)_n}{(n^2+1-n)_n} \cdot \sum_{k=0}^n \frac{c_k^2}{2k+1} \cdot \frac{(n^2+1-k)_k}{(n^2+2)_k} \\ &\leq \frac{\left(1 + \frac{n}{n^2+1}\right)^n}{\left(1 - \frac{n}{n^2+1}\right)^n} S_n^2 = O(1). \end{aligned}$$

3. A Weighted Hausdorff Moment Problem

Extensions of Legendre polynomials and the discrete Tchebycheff polynomials exist, so it is natural to see if they can be used to obtain an extension of Hausdorff's theorem. To this end, set

$$(3.1) \quad \mu_v = \int_0^1 f(x) x^{v+\alpha} (1-x)^\beta dx, \quad v = 0, 1, 2, \dots \quad (\alpha, \beta > -1).$$

Polynomials orthogonal with respect to $x^\alpha (1-x)^\beta$ on $[0, 1]$ are known. They are called Jacobi polynomials and are given by

$$(3.2) \quad P_n^{(\alpha, \beta)}(1-2x) = \frac{(\alpha+1)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; x \right).$$

Set

$$f(x) \sim \sum_{v=0}^{\infty} c_v P_v^{(\alpha, \beta)}(1-2x)$$

where c_v is determined by

$$(3.3) \quad c_v = \frac{1}{h_v^{(\alpha, \beta)}} \int_0^1 f(x) P_v^{(\alpha, \beta)}(1-2x) x^\alpha (1-x)^\beta dx$$

and

$$(3.4) \quad \int_0^1 P_n^{(\alpha, \beta)}(1-2x) P_k^{(\alpha, \beta)}(1-2x) x^\alpha (1-x)^\beta dx = \delta_{kn} h_n^{(\alpha, \beta)}$$

with

$$h_n^{(\alpha, \beta)} = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1) \cdot n!}.$$

As in the last section

$$\begin{aligned} \Delta^{n-v} \mu_v &= \int_0^1 f(x) x^{v+\alpha} (1-x)^{n-v+\beta} dx \\ &= \sum_{k=0}^n c_k \int_0^1 P_k^{(\alpha, \beta)}(1-2x) x^{v+\alpha} (1-x)^{n-v+\beta} dx \\ &= \frac{\Gamma(v+\alpha+1)\Gamma(n-v+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \sum_{k=0}^n \frac{(\alpha+1)_k}{k!} c_k \cdot {}_3F_2 \left(\begin{matrix} -k, k+\alpha+\beta+1, v+\alpha+1 \\ \alpha+1, n+\alpha+\beta+2 \end{matrix}; 1 \right). \end{aligned}$$

Using (2.5) gives

$$\begin{aligned} &\frac{\Gamma(n+\alpha+\beta+2) \Delta^{n-v} \mu_v}{\Gamma(v+\alpha+1)\Gamma(n-v+\beta+1)} \\ (3.5) \quad &= \sum_{k=0}^n \frac{(\alpha+1)_k}{k!} c_k \frac{(n+1-k)_k}{(n+\alpha+\beta+2)_k} \cdot {}_3F_2 \left(\begin{matrix} -k, k+\alpha+\beta+1 \\ \alpha+1, -n \end{matrix}; 1 \right). \end{aligned}$$

The general discrete Tchebycheff polynomials [6] (or to use their common name, the Hahn polynomials) are given by

$$(3.6) \quad Q_k(x; \alpha, \beta, n) = {}_3F_2 \left(\begin{matrix} -k, k+\alpha+\beta+1, -x \\ \alpha+1, -n \end{matrix}; 1 \right), \quad (k, x = 0, 1, \dots, n).$$

Their orthogonality relation is

$$(3.7) \quad \sum_{x=0}^n Q_k(x; \alpha, \beta, n) Q_j(x; \alpha, \beta, n) \binom{x+\alpha}{x} \binom{N-x+\beta}{N-x} \\ = \delta_{jk} \cdot \frac{(\alpha + \beta + 2)_n \cdot k! (n + \alpha + \beta + 2)_k (\beta + 1)_k (\alpha + \beta + 1)_k}{n! (n + 1 - k)_k (\alpha + 1)_k (\alpha + \beta + 1)_k (2k + \alpha + \beta + 1)_k},$$

$$0 \leq j, k \leq n.$$

Square (3.5), multiply by $\binom{v+\alpha}{v} \binom{n-v+\beta}{n-v}$ and sum. After simplification, the resulting identity is

$$\sum_{v=0}^n |\Delta^{n-v} \mu_v|^2 \binom{n}{v} \frac{\Gamma(n + \alpha + \beta + 2)}{\Gamma(v + \alpha + 1) \Gamma(n - v + 1)} \cdot \sum_{k=0}^n c_{k,k}^{2h(\alpha, \beta)} \cdot \frac{(n + 1 - k)_v}{(n + \alpha + \beta + 2)_k}.$$

The Riesz-Fisher theorem for Jacobi series is

$$\int_0^1 (f(x))^2 x^\alpha (1-x)^\beta dx = \sum_{k=0}^{\infty} c_{k,k}^{2h(\alpha, \beta)}$$

so an argument similar to the one in §2 gives the following:

Theorem 1. Define μ_v by (3.1). Then for $\alpha, \beta > -1$,

$$\int_0^1 |f(x)|^2 x^\alpha (1-x)^\beta dx < \infty$$

if and only if

$$\sum_{v=0}^n |\Delta^{n-v} \mu_v|^2 \binom{n}{v} \frac{\Gamma(n + \alpha + \beta + 2)}{\Gamma(v + \alpha + 1) \Gamma(n - v + \beta + 1)} = O(1).$$

This can be rephrased as

$$\int_0^1 |f(x) x^\alpha (1-x)^\beta|^2 \frac{dx}{x^\alpha (1-x)^\beta} < \infty$$

if and only if

$$(n+1) \sum_{v=0}^n |\Delta^{n-v} \mu_v|^2 \binom{n}{v}^2 \frac{1}{\left(\frac{v+\frac{1}{2}}{n+1}\right)^\alpha \left(1 - \frac{v+\frac{1}{2}}{n+1}\right)^\beta} = O(1),$$

when $\alpha, \beta > -1$.

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